

# Relativistically covariant state-dependent cloning of photons

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The influence of the relativistic covariance requirement on the optimality of the symmetric state-dependent  $1 \rightarrow 2$  cloning machine is studied. Namely, given a photonic qubit whose basis is formed from the momentum-helicity eigenstates, the change to the optimal cloning fidelity is calculated taking into account the Lorentz covariance unitarily represented by Wigner’s little group. To pinpoint some of the interesting results, we found states for which the optimal fidelity of the cloning process drops to  $2/3$  which corresponds to the fidelity of the optimal classical cloner. Also, an implication for the security of the BB84 protocol is analyzed.

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## I. INTRODUCTION

In recent years the influence of special and general relativity on quantum information processing has begun to be investigated [1]. Staying just within the realm of special relativity, one of the natural questions is how Lorentz transformations affect the properties of both massive and massless particle states. The basic approach is through Wigner’s little group machinery [2] as the Poincare group is non-compact and thus without any finite-dimensional unitary representations [3, 4]. In this context, the entanglement properties of bipartite states [5] composed of massive as well as massless particles have been extensively analyzed. The fundamental impact on entanglement when the owners of both entangled subsystems are Lorentz transformed [6] has been recognized since, for physically plausible states, entanglement depends on the properties of the frame where it is measured [7, 8].

In this paper, we go back to single-qubit transformations and discuss quantum cloning from the relativistic point of view. We study how the requirement of Lorentz covariance affects the optimality of a cloning protocol. Quantum cloning has come a long way since the discovery of the no-cloning theorem [9] and one can find an extensive variety of cloners in two recent review articles [10]. Lorentz covariance means that the particular cloning map must be equally effective irrespective of how any input state is rotated or boosted. More precisely, choosing the fidelity between an input and output state as a figure of merit to measure the quality of the clones, we demand that its value be maximal and independent on the input qubit. The additional requirement of maximality provides an optimal cloner. As a striking example of how the relativistic covariance constraint modifies the optimality of the fidelity results, we investigate the state dependent  $1 \rightarrow 2$  cloner of, generally non-orthogonal, qubits presented in Ref. [11]. In the relativistic domain it is necessary to distinguish between particular particle states for which the effect of the little group generally differ. Photons polarization defines the logical basis of qubits and it is a natural choice due to their use in quantum communication protocols such as BB84. There are relatively few previous studies devoted to relativistic effects in either classical or quantum channels. For the classical ones we highlight Ref. [12] where the channel capacity between two moving observers is studied. For quantum channels, there has been a recent growing interest on quantum information processing in black holes [13] as well as on how the Unruh effect assists quantum state encryption [14].

## II. WIGNER PHASE AND PHOTONIC WAVE PACKETS

As usual, a standard momentum light-like 4-vector  $k^\mu$  with  $k^0 > 0$  and  $k_\mu k^\mu = 0$  is chosen. We can transform this vector into an arbitrary light-like 4-vector  $p^\nu = L(p)^\nu_\mu k^\mu$  by a standard Lorentz transformation. The most general little group element (stabilizer subgroup which leaves  $k^\nu$  invariant) is  $W(\Lambda, p) = L_{\Lambda p}^{-1} \Lambda L_p$  [3] and consists of rotations and/or translations in a plane, a group which is isomorphic to the Euclidean group ( $ISO(2)$ ).

The corresponding Hilbert space is spanned by vectors with two indices since, together with the angular momentum  $\hat{\mathbf{J}}$ , the translation operator of the Poincare group  $\hat{\mathbf{P}}$  yields the complete set of commuting operators. In any given reference frame, a rotation around the direction defined by the standard vector induces a phase on the corresponding state in Hilbert space:

$$e^{-i\gamma\hat{J}_3}|k; \sigma\rangle = e^{-i\sigma\gamma}|k; \sigma\rangle, \quad (1)$$

where for simplicity we took  $k_\mu = \omega_0(1, 0, 0, 1)$  and  $\sigma$  is the component of the angular momentum in the direction of  $k_\mu$  (helicity). As is well known [3], massless particles have only integer or semi-integer  $\sigma$  values and for photons  $\sigma = \pm 1$  holds.

Any other state is obtained from the reference state  $|k; \sigma\rangle$  by applying a standard Lorentz transformation. Thus, if  $p_\mu$  is obtained from  $k_\mu$  by a rotation with longitudinal angle  $\theta$  and azimuthal angle  $\varphi$ , then

$$|p; \sigma\rangle = e^{-i\varphi \hat{J}_3} e^{-i\theta \hat{J}_2} e^{i\varphi \hat{J}_3} |k; \sigma\rangle \quad (2)$$

Transforming this state vector by an arbitrary Lorentz transformation  $\Lambda$  we arrive at the unitary representation, which turns out to be one-dimensional  $D(W(\Lambda, p)) = \exp(i\sigma\vartheta_W(\Lambda, p))$ . Here  $\vartheta_W(\Lambda, p)$  is an angle of the rotation dependent on the Lorentz transformation  $\Lambda$  and the initial 4-vector  $p$  (the explicit form of  $\vartheta_W$  can be found in [15, 16]). Then, we have

$$U(\Lambda) |p, \sigma\rangle = \exp(i\sigma\vartheta_W) |\Lambda p, \sigma\rangle. \quad (3)$$

We suppose that a wave packet is prepared in a state

$$|\Psi_f\rangle = \int \sum_{\sigma=\pm 1} d\mu(p) f_\sigma(p) a_{p,\sigma}^\dagger |vac\rangle, \quad (4)$$

where  $d\mu(p)$  is a Lorentz-invariant measure and  $f_\sigma(p)$  is a normalized weight function  $\int \sum_\sigma d\mu(p) |f_\sigma(p)|^2 = 1$  that describes the superposition of modes with different frequencies but a common direction of propagation  $p$ . This selection is made to avoid problems coming from diffraction effects occurring for a general wave packet so that one cannot simply define a polarization matrix [29].

Let us examine the action of an arbitrary Lorentz transformation on  $|\Psi_f\rangle$ . We know [16] that the phase angle does not depend on the magnitude of  $p$  but just on its direction. So making the transformation  $\Lambda |\Psi_f\rangle$  the phase  $\exp(i\sigma\vartheta_W(\Lambda, p))$  is common for the whole wave packet. Considering the choice of our wave packet and also the discussion in [17], after the Lorentz transformation and tracing over the momenta degree of freedom, we get

$$|\Lambda \Psi_f\rangle = \int d\mu(p) \sum_{\sigma=\pm 1} e^{i\sigma\vartheta_W} f_\sigma(\Lambda p) |\Lambda p, \sigma\rangle \xrightarrow{\text{Tr}_p} \varrho = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* e^{2i\vartheta_W} \\ \alpha^* \beta e^{-2i\vartheta_W} & |\beta|^2 \end{pmatrix}, \quad (5)$$

where  $|\alpha|^2 = \int d\mu(p) |f_1(\Lambda p)|^2$ ,  $|\beta|^2 = \int d\mu(p) |f_{-1}(\Lambda p)|^2$  and  $\alpha\beta^* = \int d\mu(p) f_1(\Lambda p) f_{-1}^*(\Lambda p)$ . The helicity basis is the logical basis  $\{|0\rangle, |1\rangle\}$  for our qubits (we thus do not use the Lorentz invariant logical basis composed of two physical photons proposed in [18] - the task is to clone an unknown single-photon state).

### III. RELATIVISTICALLY COVARIANT CLONING

For the rest of the article, we assume the following spacetime arrangement. In her reference frame, Alice prepares a state which travels in the  $p$ -direction. Although this direction is well-defined by the outgoing state, for a subject in another inertial reference frame who receives the state (Bob) it is not sufficient information. The reason is that there is the whole group of transformations (rotations around the  $p$ -direction) which leaves the given light-like vector intact. This is exactly the 'rotational' part of Wigner's little group responsible for inducing the Wigner phase  $\vartheta_W$  in (3) and both angles (rotation and Wigner phase) coincide [16]. We consider Bob's rotation to be completely unknown and uniformly distributed.

Let us proceed to analyze how the state-dependent cloning setup investigated by Bruß *et al.* [11] is affected if relativistic covariance is incorporated. First, let us remember the original problem and later we formulate how relativistic covariance enters the game. From now on  $\sigma_X, \sigma_Y$  and  $\sigma_Z$  denote the Pauli  $X, Y$  and  $Z$  matrices.

The original problem solved in [11] is, in some sense, an opposite extreme compared to the universal cloner [26], where all possible pure states are distributed according to the Haar measure. Here, for a fixed  $\xi$ , one of just two real states  $\{|\psi\rangle = \cos(\xi/2)|0\rangle + \sin(\xi/2)|1\rangle, \sigma_X |\psi\rangle\}, \xi \in (0, \pi/2)$  is prepared at random. The authors solved the problem assuming several reasonable invariance constraints. The output states were symmetric with respect to the bit flip  $\sigma_X$  and were also permutationally invariant. For later comparison with our results, the fidelity function obtained in Ref. [11] is shown in Fig. 1. The angles of the qubit parametrization are re-scaled to conform the parametrization used here.

Now we evaluate (and later generalize) the same setup when the relativistic covariance condition is imposed. Eq. (5) describes the effect of Lorentz transformation on the photonic states that are considered here. On the Bloch sphere,

this transformation becomes  $P_{\vartheta_W} = \exp(i\vartheta_W/2(\mathbb{1} - \sigma_Z))$ . Since our requirement is that the actual angle of the rotation is unknown and uniformly distributed so are the states on the Bloch sphere. Hence, in addition to the symmetries described in the previous paragraph, we require the invariance of the output with respect to the operator  $P_{\vartheta_W}$ . The invariance reflects the ignorance of the rotation angle that induces the phase angle  $\vartheta_W$  in Eq. (5).

Let us pause here and describe the physical situation. We suppose that Alice's covariant operation is  $\sigma_X$  (plus some additional operations which we won't mention again) which is combined with another covariant operation  $P_{\vartheta_W}$  induced by the Wigner phase  $\vartheta_W$  (that is, Alice sends one of two possible states which could be transformed by Bob's rotation) so we need to compare the action of  $P_{\vartheta_W}$  and  $P_{\vartheta_W}\sigma_X$ . The order is important because the operators do not commute. It can be easily shown that for single qubits,

$$P_{\vartheta_W}\sigma_X = \sigma_X P_{-\vartheta_W} \quad (6)$$

what will prove to be very useful for later calculations.

If we want to go beyond the setup studied in [11] and suppose that Alice may prepare a general pure qubit in the form  $|\psi\rangle_{gen} = \cos(\xi/2)|0\rangle + e^{i\phi}\sin(\xi/2)|1\rangle$ ,  $\xi \in (0, \pi/2)$ ,  $\phi \in (0, 2\pi)$  we find that  $P_{\vartheta_W}$  and  $P_{\vartheta_W}\sigma_X$  (our covariant operations) have a curious behavior since when they are applied to  $|\psi\rangle_{gen}$  these transformations appear in general as two asymmetric oriented arcs on opposite hemispheres (parametrized by  $\vartheta_W$ ). To get a symmetric relativistic transformation we have to assume a different covariant operation, namely  $\text{Ad}(P_{\vartheta_W})\Gamma\text{Ad}(\sigma_X)$  where  $\text{Ad}(U)[\varrho] = U\varrho U^{-1}$  is the conjugation operation so we are in an adjoint representation of a group whose members are  $U$  [20].  $\Gamma$  is the transposition of the density matrix in the standard (logical) basis  $\varrho \xrightarrow{\Gamma} \varrho^T$  (because of this transformation we traveled into the adjoint representation). The reason for incorporating  $\Gamma$  becomes evident when we compare the action of  $\text{Ad}(P_{\vartheta_W})$  and  $\text{Ad}(P_{\vartheta_W})\Gamma\text{Ad}(\sigma_X)$  (our new covariant couple) on  $|\psi\rangle_{gen}$ . In this case, we will make use of the following identity (see proof in Appendix)

$$\text{Ad}(P_{\vartheta_W})\Gamma\text{Ad}(\sigma_X) = \Gamma\text{Ad}(\sigma_X P_{\vartheta_W}). \quad (7)$$

Note that  $[\Gamma, \text{Ad}(\sigma_X)] = 0$ . The motivation for introducing the identity is purely computational (just as for commutator (6)) but the physical interpretation is interesting. Since  $[\text{Ad}(P_{\vartheta_W}), \Gamma\text{Ad}(\sigma_X)] = 0$  holds the order of the covariance operations does not matter. Thus, in the next we will investigate both relativistic covariance effects, i.e. when covariance is required with respect to  $\text{Ad}(P_{\vartheta_W}\sigma_X)$  for real states  $\{|\psi\rangle\}$ , and  $\text{Ad}(P_{\vartheta_W})\Gamma\text{Ad}(\sigma_X)$  for general states  $\{|\psi\rangle_{gen}\}$ . Except where really necessary, we will omit the symbol  $\text{Ad}()$  for the conjugation operation to avoid the excessive notation but we have to remember that we keep working in the adjoint representation.

Let us rephrase the invariance requirements from the previous paragraph in an appropriate formalism. The Jamiołkowski isomorphism [21] between positive operators and CP maps [22] is a traditional tool for the calculation of optimal and group covariant completely positive (CP) maps. One appreciates the representation even more by realizing that an implementation of the mentioned transposition operation is particularly easy. Let  $\mathcal{M}$  be a CP map, then the corresponding positive operator  $R_{\mathcal{M}}$  is related by

$$\mathcal{M}^{(N)}(\varrho_{in}) = \text{Tr}_{in} \left[ (\mathbb{1} \otimes \Gamma^{\circ N}[\varrho_{in}]) R_{\mathcal{M}}^{(N)} \right], \quad (8)$$

with  $N = 1, 2$  denoting the above discussed alternatives (without and with the transposition, respectively) and  $\Gamma^{\circ 1} \equiv \Gamma$ ,  $\Gamma^{\circ 2} = \Gamma \circ \Gamma = \mathbb{1}$ . The expression  $\Gamma^{\circ 1}[\varrho_{in}] \equiv \varrho_{in}^T$  stands for the transposition of the density matrix  $\varrho_{in}$ . It is important to stress that the case  $N = 2$  must not be in a contradiction with the definition of the isomorphism ( $N = 1$ ). Consequently, the net effect is that we require  $R_{\mathcal{M}}^{(2)}$  to be invariant with respect to the transposition of  $\varrho_{in}$ .

If  $\mathcal{M}$  is a cloning CP map then using Eq. (6) for  $N = 1$  and Eq. (7) for  $N = 2$  we can first start by requiring covariance with respect to  $P_{\mp\vartheta_W}$ . Then the covariance conditions in both representations (standard and Jamiołkowski, respectively) read

$$\mathcal{M}^{(1)}(\varrho) = (P_{-\vartheta_W} \otimes P_{-\vartheta_W})^\dagger \mathcal{M} \left( P_{-\vartheta_W} \varrho P_{-\vartheta_W}^\dagger \right) (P_{-\vartheta_W} \otimes P_{-\vartheta_W}) \rightleftharpoons [R_{\mathcal{M}}^{(1)}, P_{-\vartheta_W} \otimes P_{-\vartheta_W} \otimes P_{-\vartheta_W}^*] = 0 \quad (9a)$$

$$\mathcal{M}^{(2)}(\varrho) = (P_{\vartheta_W} \otimes P_{\vartheta_W})^\dagger \mathcal{M} \left( P_{\vartheta_W}^* \varrho^T P_{\vartheta_W}^T \right) (P_{\vartheta_W} \otimes P_{\vartheta_W}) \rightleftharpoons [R_{\mathcal{M}}^{(2)}, P_{\vartheta_W} \otimes P_{\vartheta_W} \otimes P_{\vartheta_W}] = 0. \quad (9b)$$

Note that  $P_{\vartheta_W}^* = P_{\vartheta_W}^\dagger = P_{-\vartheta_W}$ . Let us explain the use of the Jamiołkowski isomorphism. For  $N = 1$ , utilizing Eq. (6) we apply the covariance condition coming from the structure of the phase operator  $P_{-\vartheta_W}$ . We get the basic structure of the Jamiołkowski operator  $R_{\mathcal{M}}^{(1)}$  and we apply the bit-flip and the output state symmetry covariance conditions. Similarly for  $N = 2$ , the strategy is to summon the rhs of Eq. (7) to find how the covariance condition coming from  $P_{\vartheta_W}$  defines the basic structure of  $R_{\mathcal{M}}^{(2)}$ . Then, in addition to the previously mentioned covariant conditions, we

require the covariance regarding the transposition of an input state. This is the reason why on the lhs of Eq. (9b) there is  $\Gamma \circ [P_{\vartheta_W} \varrho P_{\vartheta_W}^\dagger] = P_{\vartheta_W}^* \varrho^T P_{\vartheta_W}^T$ . Finally, we calculate single-copy fidelities of the cloned state for both cases. The operator  $R_{\mathcal{M}}^{(1)}$  is just a unitary modification of  $R_{\mathcal{M}}^{(2)}$  as seen from Eqs. (9) so it is sufficient to analyze the structure of the case  $N = 2$  and for  $N = 1$  to subsequently modify the operator by  $\sigma_Y \otimes \sigma_Y \otimes \mathbb{1}$  – it is a simple permutation of basis states.

One of the Schur lemmas gives us the structure of the positive operator  $R_{\mathcal{M}}^{(2)}$ . It is a sum of the isomorphisms between all equivalent irreducible representations, which in the case of  $P_{\vartheta_W} \in U(1)$  are all one-dimensional and are distinguished by the character values  $e^{in\vartheta_W}$  with  $n \in \mathbb{Z}$ . More specifically,  $P_{\vartheta_W} \otimes P_{\vartheta_W} \otimes P_{\vartheta_W}$  is composed of four irreducible representations. Two of them are one dimensional (spanned by  $\{|0\rangle\}, \{|7\rangle\}$ ) and two are three dimensional  $\{|1\rangle, |2\rangle, |4\rangle\}, \{|3\rangle, |5\rangle, |6\rangle\}$  where  $|m\rangle$  is a decimal record of the 3-qubit basis.

Taking into account the above discussed additional symmetries of  $R_{\mathcal{M}}^{(1,2)}$  the number of independent parameters gets limited and we arrive to the following form of  $R_{\mathcal{M}}^{(2)}$

$$\begin{aligned} R_{\mathcal{M}}^{(2)} = & c_{00}(|0\rangle\langle 0| + |7\rangle\langle 7|) + c_{11}(|1\rangle\langle 1| + |6\rangle\langle 6|) + c_{22}(|2\rangle\langle 2| + |5\rangle\langle 5|) + c_{33}(|3\rangle\langle 3| + |4\rangle\langle 4|) \\ & + c_{24}(|2\rangle\langle 4| + |4\rangle\langle 2| + |3\rangle\langle 5| + |5\rangle\langle 3|) \\ & + c_{12a}(|1\rangle\langle 2| + |1\rangle\langle 4| + |3\rangle\langle 6| + |5\rangle\langle 6| + h.c.) + i c_{12b}(|1\rangle\langle 2| + |1\rangle\langle 4| + |3\rangle\langle 6| + |5\rangle\langle 6| - h.c.), \end{aligned} \quad (10)$$

where  $c_{ij} \in \mathbb{C}$  (for  $i \neq j$ ) are coefficients of the isomorphisms  $|i\rangle\langle i| \leftrightarrow |j\rangle\langle j|$  and  $c_{12a} = \Re[c_{12}], c_{12b} = \Im[c_{12}]$ . Two additional conditions come from the trace-preserving constraint  $\text{Tr}_{out} [R_{\mathcal{M}}^{(N)}] = \mathbb{1} \Rightarrow c_{00} + c_{11} + c_{22} + c_{33} = 1$  (common for both  $N$ ) and, of course, from the positivity condition  $R_{\mathcal{M}}^{(N)} \geq 0$ .

Since we are cloning a pure qubit, our figure of merit to be maximized is the single copy fidelity between the input states  $|\psi\rangle = \cos(\xi/2)|0\rangle + \sin(\xi/2)|1\rangle$  (for  $N = 1$ ) and  $|\psi\rangle_{gen} = \cos(\xi/2)|0\rangle + e^{i\phi}\sin(\xi/2)|1\rangle$  (for  $N = 2$ ) and the target states of the same form

$$F^{(1)} = \text{Tr} \left[ (|\psi\rangle\langle\psi| \otimes \mathbb{1} \otimes \Gamma [|\psi\rangle\langle\psi|]) R_{\mathcal{M}}^{(1)} \right] \quad (11a)$$

$$F^{(2)} = \text{Tr} \left[ (|\psi\rangle\langle\psi|_{gen} \otimes \mathbb{1} \otimes |\psi\rangle\langle\psi|_{gen}) R_{\mathcal{M}}^{(2)} \right]. \quad (11b)$$

Observe that in Eq. (11b) the transposition operator was additionally applied.

**Case  $N = 1$  (covariance w.r.t.  $P_{\vartheta_W}$  and  $P_{\vartheta_W} \sigma_X$ )** If we apply the covariant operations on an arbitrary real  $|\psi\rangle$  then for different values of the Wigner phase  $\vartheta_W$  we generate two symmetric trajectories on the opposite hemispheres of the Bloch sphere. Reformulating the search for the fidelity as a semidefinite program using the SeDuMi solver [24] in the YALMIP environment [25] the number of parameters is reduced and  $R_{\mathcal{M}}^{(1)}$  can be diagonalized. This leads to the full analytical derivation of the fidelity function (11a) as a function of the input state  $|\psi\rangle$

$$F^{(1)} = \frac{1}{2} \left[ 1 + \frac{1}{2} \cos^2 \xi \left( 1 + \frac{\cos^2 \xi}{\sqrt{2 \sin^4 \xi + \cos^4 \xi}} \right) + \frac{\sin^4 \xi}{\sqrt{2 \sin^4 \xi + \cos^4 \xi}} \right]. \quad (12)$$

The function is depicted in Fig. 1 and we notice several interesting things. Obviously, the fidelity is lower than the original state dependent fidelity. We observe that the minimum moved from  $\xi_{min}^{\text{BruB}} = \pi/6$  [11] to the angle  $\xi_{min} = \text{arccot} \sqrt{1/2}$  but more interesting point is that the fidelity attains  $F_{min}^{(1)} = 5/6$ . This value is 'reserved' for the  $1 \rightarrow 2$  universal symmetric cloner [26], i.e. the cloning map covariant with respect to the action of  $SU(2)$  (or, equivalently, to the cloning of all mutually unbiased states of the Bloch sphere [10]). Such a low value for a kind of phase-covariant cloner we are investigating may be surprising. For  $\xi = \pi/2$  we recover the result from [23] where  $F = \frac{1}{2} + \sqrt{\frac{1}{8}}$ . This is expected because the bit flip (one of our additional conditions) is unnecessary on the equator (due to the presence of  $P_{\vartheta_W}$ ).

**Case  $N = 2$  (covariance w.r.t.  $P_{\vartheta_W}$  and  $P_{\vartheta_W} \Gamma \circ \sigma_X$ )** Using methods similar to those in the previous paragraph we arrive with the help of Eq. (11b) and  $|\psi\rangle_{gen}$  at the following form of the fidelity function

$$F^{(2)} = \max \left\{ \frac{1}{4}(\cos 2\xi + 3), \frac{1}{2} \left[ 1 + \frac{1}{2} \cos^2 \xi \left( -1 + \frac{\cos^2 \xi}{\sqrt{2 \sin^4 \xi + \cos^4 \xi}} \right) + \frac{\sin^4 \xi}{\sqrt{2 \sin^4 \xi + \cos^4 \xi}} \right] \right\}, \quad (13)$$

which is independent on the input state phase  $\phi$ . This result is no less interesting and the function is again depicted in Fig. 1. The minimum angle is common with the previous case but the corresponding fidelity drops to  $F_{min}^{(2)} = 2/3$ .

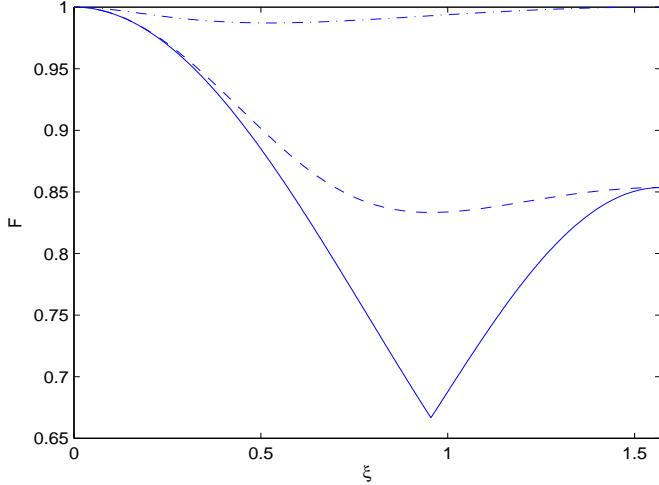


FIG. 1: Illustration of how the local fidelity of a state dependent  $1 \rightarrow 2$  symmetric qubit cloner studied in [11] (dash-dotted line) changes when some additional symmetries stemming from the relativistic covariance are required. First, a state dependent phase-covariance is added (dashed line) and the minimal fidelity  $F_{min}^{(1)} = 5/6$  is reached for  $\xi_{min} = \text{arccot } \sqrt{1/2}$ . Furthermore, the transposition transformation corresponding to the finding of an orthogonal complement is considered and for the same  $\xi_{min}$  the minimal fidelity (solid line) reaches  $F_{min}^{(2)} = 2/3$ .

This low value can be justified if we realize what kind of operation corresponds to  $N = 2$ . We combine two impossible operations, quantum cloning and finding the universal-NOT operation, into what is together known as the anti-cloning operation [27]. This combined requirement is apparently stronger than the universal (i.e.  $SU(2)$ ) covariance and the reason for the low fidelity values is that the map  $R_{\mathcal{M}}^{(2)}$  must be of the same form for both  $\varrho$  and  $\varrho^T$  as a result of Eq. (8).

Could this result tell us something about, for instance, the security of quantum key distribution (QKD)? Looking at the most studied protocol BB84 [28] (of course, implemented by the polarization encoding which is preferred for a free-space communication for which the relativistic effects may be very relevant) we see that four qubits equidistantly distributed on the meridian are used. For  $N = 1$ , they form the  $xz$ -plane of the Bloch sphere ('real meridian') and for  $N = 2$  it is an arbitrary grand circle intersecting the north and south pole ('complex meridian'). From the viewpoint of an eavesdropper without the knowledge of the Wigner phase  $\vartheta_W$  and decided to clone the quantum states to get some information, we can now demonstrate that not all quadruples are equally good. If the states  $\{\cos(\pi/8)|0\rangle \pm \sin(\pi/8)|1\rangle, \cos(\pi/8)|1\rangle \pm \sin(\pi/8)|0\rangle\}$  are used for the QKD purposes then by inserting  $\xi = \pi/4$  into Eqs. (12) and (13), we get  $F^{(1)} = (5 + \sqrt{3})/8 \simeq 0.8415$  and  $F^{(2)} = 3/4$ . On the other hand, using the quadruple  $\{|0\rangle, |1\rangle, 1/\sqrt{2}|0 \pm 1\rangle\}$  we get the fidelity  $F = 5/6$  because by phase-rotating the quadruple states (that is, applying  $P_{\vartheta_W}$ ) we pass the mutually unbiased states of the Bloch sphere. We see that  $F^{(2)} < F < F^{(1)}$  corresponds to the fact that for  $N = 2$  the eavesdropper has less information about the input state.

Another interesting question is how the relativistic covariance affects the optimality of the universal cloner. Here the situation is different. In the analysis above we combined two covariant operations ( $\Gamma \circ \sigma_X$  and  $P_{\vartheta_W}$ ) which are not generally subsets of each other. On the other hand, as we saw, every Wigner rotation is a  $U(1)$  covariant rotation and since  $U(1) \subset SU(2)$  we may conclude that the optimality of the universal cloner will remain unchanged. Pictorially, it corresponds to the situation where Alice sends a completely unknown photon ( $SU(2)$  covariance) to Bob who, in addition, does not know how the whole Bloch sphere rigidly rotates (his rotation with respect to Alice). However, this is again a kind of  $SU(2)$  rotation.

#### IV. CONCLUSIONS

In conclusion, we investigated the role played by the requirement of relativistic covariance in the problem of the optimality of one of the most prominent forbidden quantum-mechanical process as quantum cloning. Observing that the effect of Wigner's little group can be translated into the language of so-called phase-covariant processes we studied how the effectiveness of the cloning process becomes modified. Particularly, we considered an observer in a different reference frame with no knowledge of the parameters of the reference frame where the state designated for cloning was

produced. Here we focused on the class of state-dependent cloners where the effect is especially appreciable. First, as a direct application of the relativistic considerations on the cloning setup studied in [11] where one of two real states is prepared in one inertial frame and cloned in another inertial frame whose transformation properties regarding the first one are completely unknown. Second, we went beyond this setup and supposed that in the first frame two general pure qubits related by a common action of the Pauli  $X$  matrix and the density matrix transposition operator might be prepared. Again, we wanted such a state to be cloned in another inertial frame without knowledge of which state was actually sent and how it was relativistically transformed. In both cases, we brought analytical expressions for local fidelities of the output states asking the fidelity to be maximal and optimal. One of the intriguing results is that in the second case the fidelity drops even below the universal cloner limit. The reason is that we combined the mentioned cloning invariance conditions with another forbidden process - finding the orthogonal complement of an unknown state. Note that even without the relativistic context we generalized the previous research on the phase-covariant cloning maps and at the same time we studied optimal covariant processes considering covariance operations which do not commute.

As an example of the consequences for communication security issues we have shown that for an eavesdropper determined to get some information by cloning a BB84 quadruple of states, not all possibilities are equally good and some provide him with more information.

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### APPENDIX A

In the following we use some of the basic properties of Lie groups [20].

Let  $A, B, C$  be invertible linear transformations. We define the conjugation operation  $c(B)[\rho] \equiv \text{Ad}(B)[\rho] = B\rho B^{-1}$  (and similarly  $c(C)$ ) satisfying  $[A, c(B)] = 0, [A, c(C)] \neq 0, [c(B), c(C)] \neq 0$ . Then if  $A^2 = c(B^2)$  we have  $A c(CB) - c(BC)A = 0$ .

*Proof.* Noting that

$$\begin{aligned} A c(CB) - c(BC)A &= 0 \\ A^2 c(CB)A - A c(BC)A^2 &= 0 && \text{multiplied by } A \text{ from left and right} \\ A^2 c(C)D - D c(C)A^2 &= 0 && D = A c(B) = c(B)A \\ A^{-2} D c(C)A^2 &= c(C)D. && \end{aligned} \tag{A1}$$

Similarly, we get  $c(B^{-2})D c(CB^2) = c(C)D$ . Equalling these two expressions we immediately see that  $Dc(C) = A^2 c(B^{-2})D c(CB^2)A^{-2}$  holding if  $A^2 = c(B^2)$ .  $\square$

Now we identify  $A = \Gamma, c(B) = \sigma_X$  and  $c(C) = P_{\vartheta_W}$  so we have  $\Gamma P_{\vartheta_W} \sigma_X [\rho] = \sigma_X P_{\vartheta_W} \Gamma [\rho]$  and to get Eq. (7) we apply  $\Gamma$  on the equation from the left and right using the fact that  $\Gamma^{\circ 2}[\rho] = \mathbb{1}$ .

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- [29] Considering a general wave packet with a spatial distribution of momenta, a general Lorentz transformation yields an intrinsic entanglement between momenta and polarization degrees of freedom [1]. The wider angular spread the packet has the more severe the influence of this entanglement is on the definition of the polarization matrix since such an object is not even rotationally invariant [19]. The second consequence is that presumably orthogonal helicity states cannot be perfectly distinguished. If in a given frame the angular and frequency spreads satisfy  $\Delta_{ang} \ll \Delta_\omega$ , then to keep the validity of the sharp inequality we have to limit the distribution of Lorentz boosts. This restriction does not affect the phase distribution discussed here.